

Lock-in range of classical PLL with impulse signals and proportionally-integrating filter

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Abstract

In the present work the model of PLL with impulse signals and active PI filter in the signal's phase space is described. For the considered PLL the lock-in range is computed analytically and obtained result are compared with numerical simulations.

Keywords: phase-locked loop, nonlinear analysis, PLL, two-phase PLL, lock-in range, Gardner's problem on unique lock-in frequency, pull-out frequency

1. Models of classical PLL with impulse signals

Consider a physical model of classical PLL in the signals space (see Fig. 1).

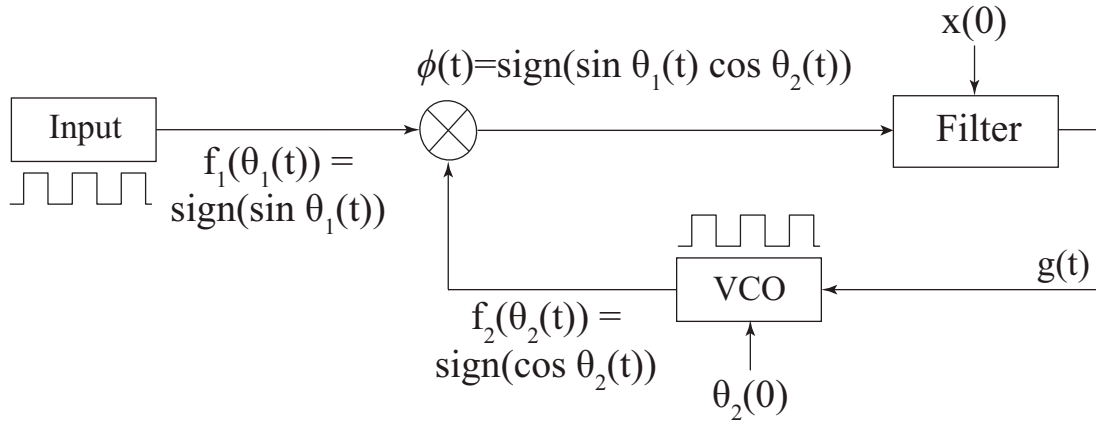


Figure 1: Model of PLL with impulse signals in the signals space.

This model contains the following blocks: a reference oscillator (Input), a voltage-controlled oscillator (VCO), a filter (Filter), and an analog multiplier as a phase detector (PD). The signals $\text{sign}(\sin \theta_1(t))$ and $\text{sign}(\cos \theta_2(t))$ of the Input and the VCO (here $\theta_2(0)$ is the initial phase of VCO) enter the multiplier block. The resulting impulse signal $\phi(t) = \text{sign}(\sin \theta_1(t) \cos \theta_2(t))$ is filtered by low-pass filter Filter (here $x(0)$ is an initial state of Filter). The filtered signal $g(t)$ is used as a control signal for VCO.

The equations describing the model of PLL-based circuits in the signals space are difficult for the study, since that equations are nonautonomous (see, e.g., (Kudrewicz

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and Wasowicz, 2007)). By contrast, the equations of model in the signal's phase space are autonomous (Gardner, 1966; Shakhgil'dyan and Lyakhovkin, 1966; Viterbi, 1966), what simplifies the study of PLL-based circuits. The application of averaging methods (Mitropolsky and Bogolubov, 1961; Samoilenko and Petryshyn, 2004) allows one to reduce the model of PLL-based circuits in the signals space to the model in the signal's phase space (see, e.g., (Leonov et al., 2012; Leonov and Kuznetsov, 2014; Leonov et al., 2015a; Kuznetsov et al., 2015b,a; Best et al., 2015)).

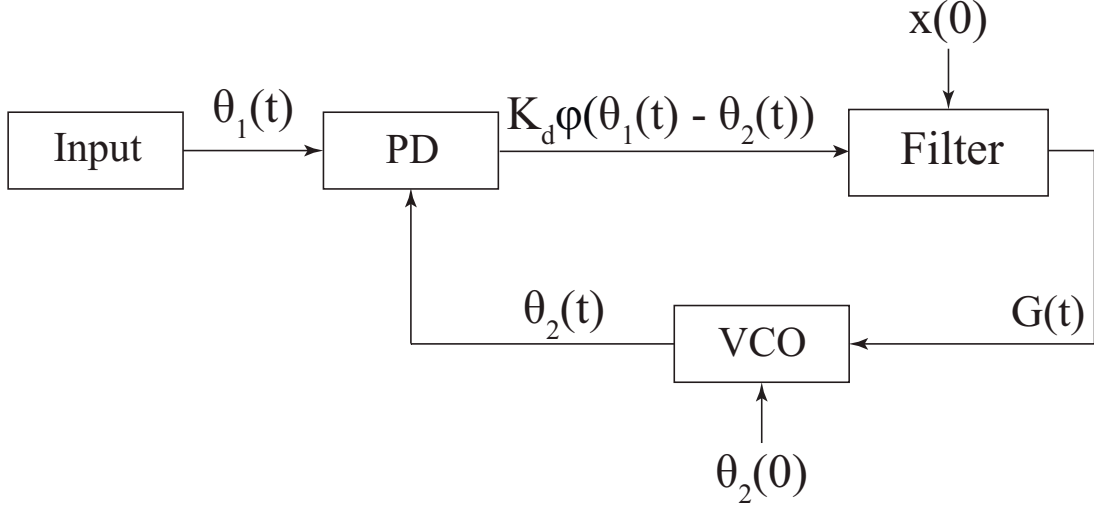


Figure 2: Model of the classical PLL in the signal's phase space.

The main difference between the physical model (Fig. 1) and the simplified mathematical model in the signal's phase space (Fig. 2) is the absence of high-frequency component of the phase detector output. The output of the phase detector in the signal's phase space is called a phase detector characteristic and has the form

$$K_d \varphi(\theta_1(t) - \theta_2(t)).$$

The maximum absolute value of PD output $K_d > 0$ is called a phase detector gain (see, e.g., (Best, 2007; Goldman, 2007)). The periodic function $\varphi(\theta_\Delta(t))$ depends on difference $\theta_1(t) - \theta_2(t)$ (which is called a phase error and denoted by $\theta_\Delta(t)$). The PD characteristic depends on the design of PLL-based circuit and the signal waveforms $f_1(\theta_1)$ of Input and $f_2(\theta_2)$ of VCO. For PLL with impulse signals the PD characteristic is as follows (see, e.g., (Viterbi, 1966; Gardner, 1966; Leonov et al., 2012)):

$$K_d = 1; \quad \varphi(\theta_\Delta(t)) = \begin{cases} \frac{2}{\pi} \theta_\Delta(t), & \text{if } -\frac{\pi}{2} \leq \theta_\Delta(t) \leq \frac{\pi}{2}, \\ -\frac{2}{\pi} \theta_\Delta(t) + 2, & \text{if } \frac{\pi}{2} \leq \theta_\Delta(t) \leq \frac{3\pi}{2}. \end{cases} \quad (1)$$

Let us describe a model of classical PLL with impulse signals in the signal's phase space (see Fig. 2). A reference oscillator and a voltage-controlled oscillator generate the phases $\theta_1(t)$ and $\theta_2(t)$, respectively. The frequency of reference signal usually assumed to be constant:

$$\dot{\theta}_1(t) = \omega_1. \quad (2)$$

The phases $\theta_1(t)$ and $\theta_2(t)$ enter the inputs of the phase detector. The output of phase detector is processed by Filter. Further we consider the active PI filter (see, e.g., (Baker,

2011)) with transfer function $W(s) = \frac{1+\tau_2 s}{\tau_1 s}$, $\tau_1 > 0$, $\tau_2 > 0$. The considered filter can be described as

$$\begin{cases} \dot{x}(t) = K_d \varphi(\theta_\Delta(t)), \\ G(t) = \frac{1}{\tau_1} x(t) + \frac{\tau_2}{\tau_1} K_d \varphi(\theta_\Delta(t)), \end{cases} \quad (3)$$

where $x(t)$ is the filter state.

The output of Filter $G(t)$ is used as a control signal for VCO:

$$\dot{\theta}_2(t) = \omega_2^{\text{free}} + K_v G(t), \quad (4)$$

where ω_2^{free} is the VCO free-running frequency and $K_v > 0$ is the VCO gain.

Relations (2), (3), and (4) result in autonomous system of differential equations

$$\begin{cases} \dot{x} = K_d \varphi(\theta_\Delta), \\ \dot{\theta}_\Delta = \omega_1 - \omega_2^{\text{free}} - \frac{K_v}{\tau_1} (x + \tau_2 K_d \varphi(\theta_\Delta)). \end{cases} \quad (5)$$

Denote the difference of the reference frequency and the VCO free-running frequency $\omega_1 - \omega_2^{\text{free}}$ by $\omega_\Delta^{\text{free}}$. By the linear transformation $x \rightarrow K_d x$ we have

$$\begin{cases} \dot{x} = \varphi(\theta_\Delta), \\ \dot{\theta}_\Delta = \omega_\Delta^{\text{free}} - \frac{K_0}{\tau_1} (x + \tau_2 \varphi(\theta_\Delta)), \end{cases} \quad (6)$$

where $K_0 = K_v K_d$ is the loop gain. Here (6) describes the model of PLL with the impulse signals and active PI filter in the signal's phase space.

By the transformation

$$(\omega_\Delta^{\text{free}}, x, \theta_\Delta) \rightarrow (-\omega_\Delta^{\text{free}}, -x, -\theta_\Delta),$$

(6) with odd PD characteristic (1) is not changed. This property allows one to use the concept of frequency deviation

$$|\omega_\Delta^{\text{free}}| = |\omega_1 - \omega_2^{\text{free}}|$$

and consider (6) with $\omega_\Delta^{\text{free}} > 0$ only.

The PLL state for which the VCO frequency is adjusted to the reference frequency of Input is called a locked state. The locked states of the PLL correspond to the locally asymptotically stable equilibria of (6), which can be found from the relations

$$\begin{cases} \varphi(\theta_{eq}) = 0, \\ \omega_\Delta^{\text{free}} - \frac{K_0}{\tau_1} x_{eq} = 0. \end{cases}$$

Since (6) is 2π -periodic in θ_Δ , we can consider (6) in a 2π -interval of θ_Δ , $\theta_\Delta \in (-\pi, \pi]$. In interval $\theta_\Delta \in (-\pi, \pi]$ there exist two equilibria:

$$(\theta_{eq}^s, x_{eq}(\omega_\Delta^{\text{free}})) = (0, \frac{\omega_\Delta^{\text{free}} \tau_1}{K_0}) \text{ and } (\theta_{eq}^u, x_{eq}(\omega_\Delta^{\text{free}})) = (\pi, \frac{\omega_\Delta^{\text{free}} \tau_1}{K_0}).$$

As is shown below (see Appendix A) the equilibria

$$(\theta_{eq}^s + 2\pi k, x_{eq}(\omega_\Delta^{\text{free}})) = \left(2\pi k, \frac{\omega_\Delta^{\text{free}} \tau_1}{K_0} \right)$$

are locally asymptotically stable. Hence, the locked states of (6) are given by equilibria $(\theta_{eq}^s, x_{eq}(\omega_{\Delta}^{\text{free}}))$. The remaining equilibria

$$(\theta_{eq}^u + 2\pi k, x_{eq}(\omega_{\Delta}^{\text{free}})) = \left(\pi + 2\pi k, \frac{\omega_{\Delta}^{\text{free}} \tau_1}{K_0} \right)$$

are saddle equilibria (see Appendix A).

2. The lock-in range

The model of classical PLL with impulse signals and active PI filter in the signal's phase space is globally asymptotically stable (see, e.g., (Gubar', 1961; Leonov and Aleksandrov, 2015)). The PLL achieves locked state for any initial VCO phase $\theta_2(0)$ and filter state $x(0)$. So, there exist no limit cycles of the first kind, heteroclinic trajectories, and limit cycles of the second kind on the phase plane of (6) (see Fig. 3).

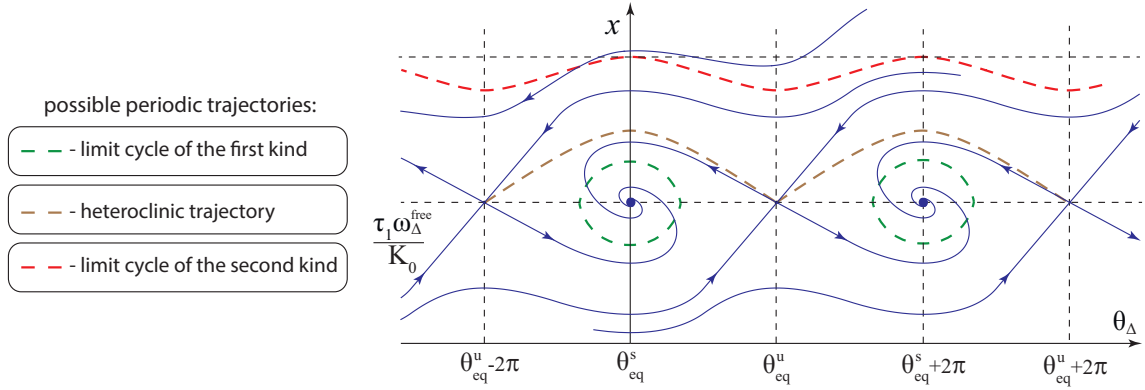


Figure 3: Possible periodic trajectories on the phase plane of (6).

However, the phase error θ_{Δ} may significantly increase during the acquisition process. In order to consider the property of the model to synchronize without undesired growth of the phase error θ_{Δ} , a lock-in range concept was introduced in (Gardner, 1966): “If, for some reason, the frequency difference between input and VCO is less than the loop bandwidth, the loop will lock up almost instantaneously without slipping cycles. The maximum frequency difference for which this fast acquisition is possible is called the lock-in frequency”. The lock-in range concept is widely used in engineering literature on the PLL-based circuits study (see, e.g., (Stensby, 1997; Kihara et al., 2002; Kroupa, 2003; Gardner, 2005; Best, 2007)). It is said that a cycle slipping occurs if (see, e.g., (Ascheid and Meyr, 1982; Ershova and Leonov, 1983; Smirnova et al., 2014))

$$\limsup_{t \rightarrow +\infty} |\theta_{\Delta}(0) - \theta_{\Delta}(t)| \geq 2\pi.$$

However, in general, even for zero frequency deviation ($\omega_{\Delta}^{\text{free}} = 0$) and a sufficiently large initial state of filter ($x(0)$), cycle slipping may take place, thus in 1979 Gardner wrote: “There is no natural way to define exactly any unique lock-in frequency” and “despite its vague reality, lock-in range is a useful concept” (Gardner, 1979).

To overcome the stated problem, in (Kuznetsov et al., 2015c; Leonov et al., 2015b) the rigorous mathematical definition of a lock-in range is suggested:

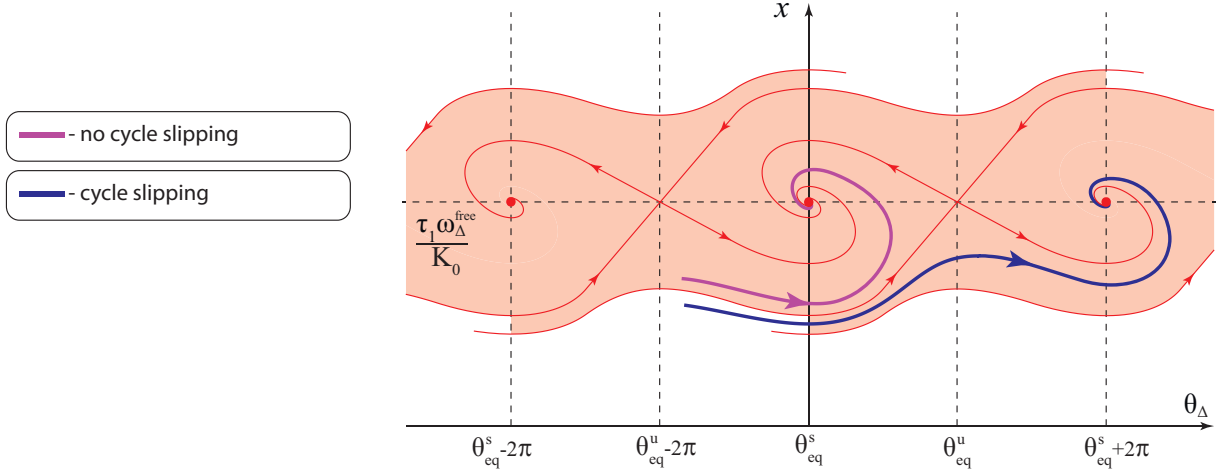


Figure 4: The lock-in domain and cycle slipping.

Definition 1. (Kuznetsov et al., 2015c; Leonov et al., 2015b) The lock-in range of model (6) is a range $[0, \omega_l)$ such that for each frequency deviation $|\omega_\Delta^{\text{free}}| \in [0, \omega_l)$ the model (6) is globally asymptotically stable and the following domain

$$D_{\text{lock-in}}((-\omega_l, \omega_l)) = \bigcap_{|\omega_\Delta^{\text{free}}| < \omega_l} D_{\text{lock-in}}(\omega_\Delta^{\text{free}})$$

contains all corresponding equilibria $(\theta_{eq}^s, x_{eq}(\omega_\Delta^{\text{free}}))$.

For model (6) each lock-in domain from intersection $\bigcap_{|\omega_\Delta^{\text{free}}| < \omega_l} D_{\text{lock-in}}(\omega_\Delta^{\text{free}})$ is bounded by the separatrices of saddle equilibria $(\theta_{eq}^u, x_{eq}(\omega_\Delta^{\text{free}}))$ and vertical lines $\theta_\Delta = \theta_{eq}^s \pm 2\pi$. Thus, the behavior of separatrices on the phase plane is the key to the lock-in range study (see Fig. 5).

3. Phase plane analysis for the lock-in range estimation

Consider an approach to the lock-in range computation of (6), based on the phase plane analysis. To compute the lock-in range of (6) we need to consider the behavior of the lower separatrix $Q(\theta_\Delta, \omega_\Delta^{\text{free}})$, which tends to the saddle point $(\theta_{eq}^u, x_{eq}(\omega_\Delta^{\text{free}})) = (\pi, \frac{\omega_\Delta^{\text{free}} \tau_1}{K_0})$ as $t \rightarrow +\infty$ (by the symmetry of the lower and the upper half-planes, the consideration of the upper separatrix is also possible). The parameter $\omega_\Delta^{\text{free}}$ shifts the phase plane vertically. To check this, we use a linear transformation $x \rightarrow x + \frac{\omega_\Delta^{\text{free}} \tau_1}{K_0}$. Thus, to compute the lock-in range of (6), we need to find $\omega_\Delta^{\text{free}} = \omega_l$ (where ω_l is called a lock-in frequency) such that (see Fig. 5)

$$x_{eq}(-\omega_l) = Q(\theta_{eq}^s, \omega_l). \quad (7)$$

By (7), we obtain an exact formula for the lock-in frequency ω_l :

$$\begin{aligned} -\frac{\omega_l}{K_0/\tau_1} &= \frac{\omega_l}{K_0/\tau_1} + Q(\theta_{eq}^s, 0). \\ \omega_l &= -\frac{K_0 Q(\theta_{eq}^s, 0)}{2\tau_1}, \end{aligned} \quad (8)$$

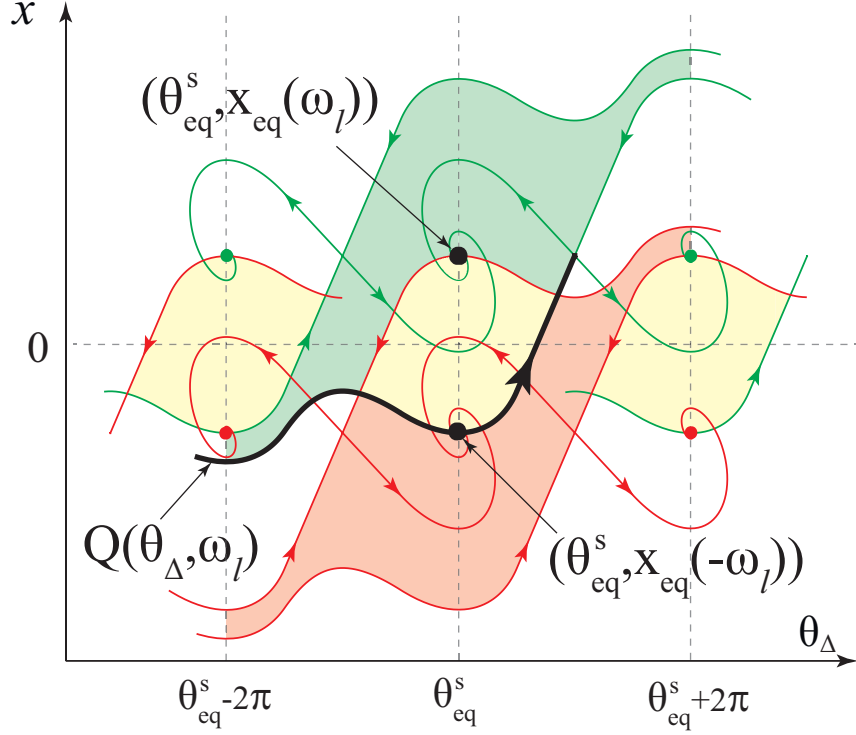


Figure 5: The lock-in domain of (6) for $|\omega_{\Delta}^{\text{free}}| = \omega_l$.

Numerical simulations are used to compute the lock-in range of (6) applying (8). The separatrix $Q(\theta_{\Delta}, 0)$ is numerically integrated and the corresponding ω_l is approximated. The obtained numerical results can be illustrated by special diagram (see Fig. 6). Note that (6) depends on the value of two coefficients $\frac{K_0}{\tau_1}$ and τ_2 . In Fig. 6, choosing X-axis as $\frac{K_0}{\tau_1}$, we can plot a single curve for every fixed value of τ_2 . The results of numerical simulations show that for sufficiently large $\frac{K_0}{\tau_1}$, the value of ω_l grows almost proportionally to $\frac{K_0}{\tau_1}$. Hence, $\frac{\omega_l \tau_1}{K_0}$ is almost constant for sufficiently large $\frac{K_0}{\tau_1}$ and in Fig. 6 the Y-axis can be chosen as $\frac{\omega_{\Delta}^{\text{free}} \tau_1}{K_0}$.

To obtain the lock-in frequency ω_l for fixed τ_1 , τ_2 , and K_0 using Fig. 6, we consider the curve corresponding to the chosen τ_2 . Next, for X-value equal $\frac{K_0}{\tau_1}$ we get the Y-value of the curve. Finally, we multiply the Y-value by $\frac{K_0}{\tau_1}$.

Consider an analytical approach to the exact lock-in range computation. Main stages of computation are presented in Subsection 3.1.

3.1. Analytical approach to the lock-in range computation

Consider a system

$$\begin{cases} \dot{\theta}_{\Delta}(t) = y(t), \\ \dot{y}(t) = -\frac{K_0 \tau_2}{\tau_1} \dot{\varphi}(\theta_{\Delta}(t)) y(t) - \frac{K_0}{\tau_1} \varphi(\theta_{\Delta}(t)), \end{cases} \quad (9)$$

where $y(t) = \omega_{\Delta}^{\text{free}} - \frac{K_0}{\tau_1} (x(t) + \tau_2 \varphi(\theta_{\Delta}(t)))$. Relations (9) are equivalent to (6) and allow one to exclude $\omega_{\Delta}^{\text{free}}$ from the computation. Note that equilibria (θ_{eq}, y_{eq}) of (9) and the corresponding equilibria (θ_{eq}, x_{eq}) of (6) are of the same type and related as

$$(\theta_{eq}, y_{eq}) = (\theta_{eq}, \omega_{\Delta}^{\text{free}} - K_0 b x_{eq}).$$

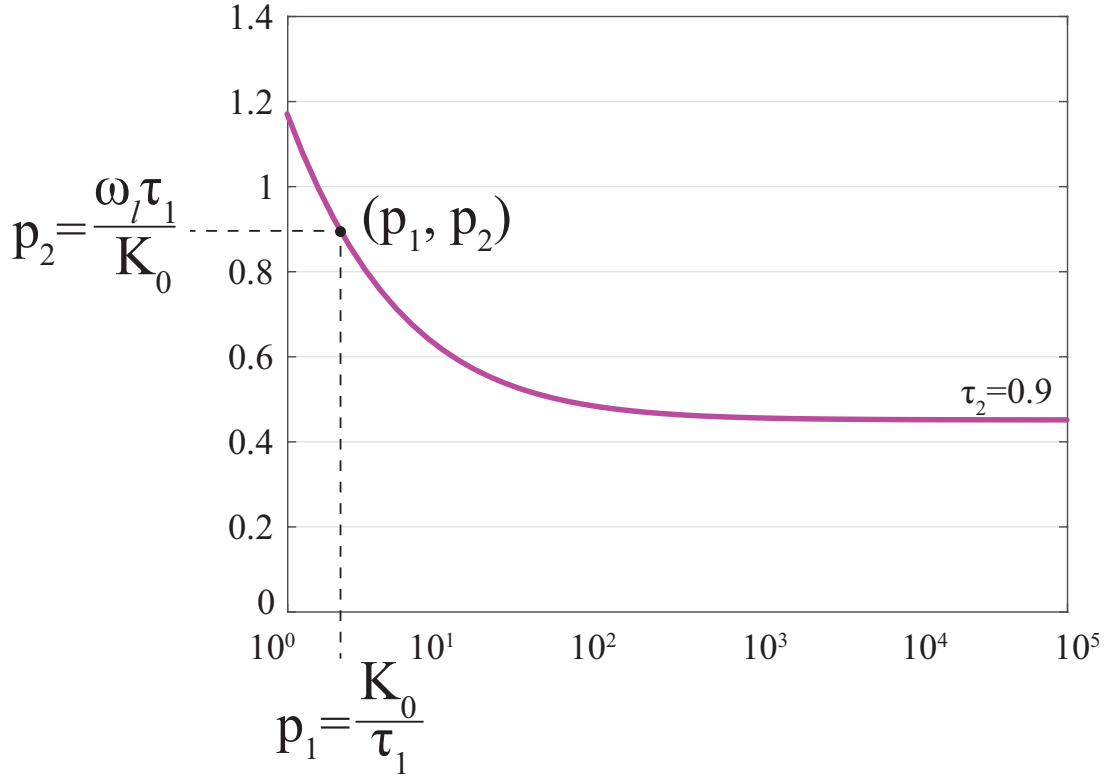


Figure 6: Diagram for the lock-in frequency ω_l calculation.

The separatrix $Q(\theta_\Delta, \omega_\Delta^{\text{free}})$ from (8) corresponds to the upper separatrix $S'(\theta_\Delta)$ of the phase plane of (9) (see Fig. 7) and the following relation

$$Q(\theta_{eq}^s, \omega_\Delta^{\text{free}}) = \frac{\tau_1}{K_0} (\omega_\Delta^{\text{free}} - S'(\theta_{eq}^s))$$

is valid.

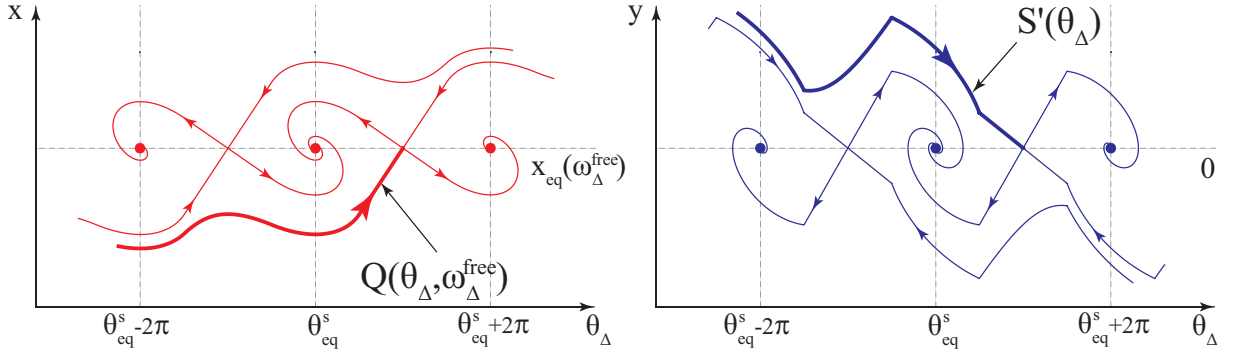


Figure 7: Phase plane portraits of (6) and (9).

Relation (8) takes the form

$$\omega_l = \frac{1}{2} S'(\theta_{eq}^s). \quad (10)$$

The computation of the separatrix $S'(\theta_\Delta)$ is in two steps. Step 1: we integrate the separatrix $S'(\theta_\Delta)$ in the interval $(\frac{\pi}{2}, \pi)$ (in which the function $\varphi(\theta_\Delta)$ is continuously differentiable) and compute $S'(\frac{\pi}{2})$. For this purpose, we need to find the eigenvector that

corresponds to separatrix $S'(\theta_\Delta)$ on the considered interval. Step 2: we find a general solution of (9) on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Here there exist three cases depending on the type stable equilibrium $(\theta_{eq}^s, 0)$: a stable focus, stable node, and stable degenerated node. For every case described above we perform separate computations. Using the computed $S'(\frac{\pi}{2})$ as the initial data of the Cauchy problem, it is possible to obtain an exact expression for $S'(\theta_{eq}^s)$.

The obtained analytical results are illustrated in Fig. 8. The red line in Fig. 8 is used for the case of stable focus, and the green line for the case of stable node. The crosses are used for the case of stable degenerated node.

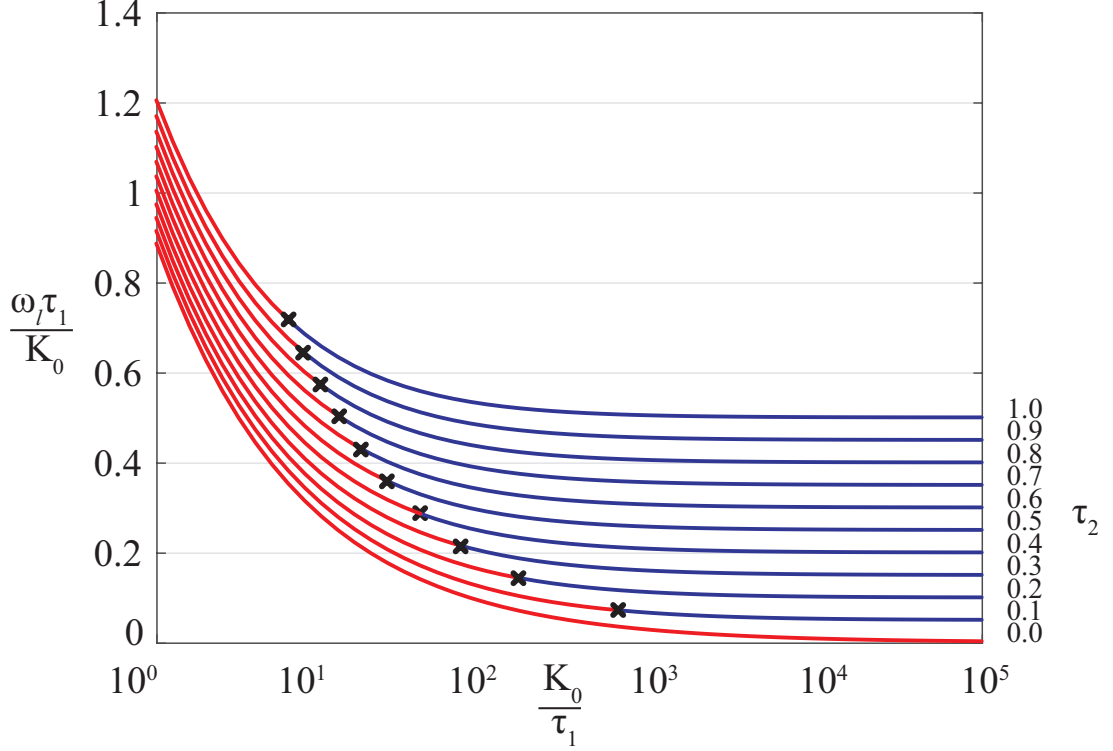


Figure 8: Diagram for the lock-in frequency ω_l calculation.

The formulae for three possible cases are given below (redefinitions $a = \frac{\tau_2}{\tau_1}$, $b = \frac{1}{\tau_1}$ are used to reduce the analytical formulae):

A. $(aK_0)^2 - 2bK_0\pi > 0$ that corresponds to a stable node:

$$\omega_l = \frac{1}{\pi} c_1 \sqrt{(aK_0)^2 - 2bK_0\pi} \left(-\frac{c_2}{c_1} \right)^{\left(\frac{1}{2} - \frac{aK_0}{2\sqrt{(aK_0)^2 - 2bK_0\pi}} \right)}, \quad (11)$$

$$\text{where } c_1 = \frac{\pi}{4} \left(\frac{\sqrt{(aK_0)^2 + 2bK_0\pi}}{\sqrt{(aK_0)^2 - 2bK_0\pi}} + 1 \right), c_2 = \frac{\pi}{4} \left(1 - \frac{\sqrt{(aK_0)^2 + 2bK_0\pi}}{\sqrt{(aK_0)^2 - 2bK_0\pi}} \right).$$

B. $(aK_0)^2 - 2bK_0\pi = 0$ that corresponds to a stable degenerated node:

$$\omega_l = \frac{1}{2} c_2 e^{\left(\frac{aK_0}{2c_2} \right)}, \text{ where } c_2 = \frac{\sqrt{(aK_0)^2 + 2bK_0\pi}}{2}. \quad (12)$$

C. $(aK_0)^2 - 2bK_0\pi < 0$ that corresponds to a stable focus:

$$\begin{aligned} \omega_l = & -\frac{aK_0 e^{t_0 \operatorname{Re} \lambda_1^s}}{2\pi} (c_1 \cos(t_0 \operatorname{Im} \lambda_1^s) + c_2 \sin(t_0 \operatorname{Im} \lambda_1^s)) + \\ & + \frac{e^{t_0 \operatorname{Re} \lambda_1^s} \sqrt{2bK_0\pi - (aK_0)^2}}{2\pi} (c_2 \cos(t_0 \operatorname{Im} \lambda_1^s) - c_1 \sin(t_0 \operatorname{Im} \lambda_1^s)), \end{aligned} \quad (13)$$

$$\text{where } t_0 = \frac{\operatorname{arctg}\left(-\frac{c_1}{c_2}\right)}{\operatorname{Im} \lambda_1^s}, c_1 = \frac{\pi}{2}, c_2 = \frac{\pi \sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})}}{2\sqrt{2bK_0\pi - (aK_0)^2}},$$

$$\lambda_1^s = \frac{-aK_0 + i\sqrt{2bK_0\pi - (aK_0)^2}}{\pi}.$$

Rigorous derivation of (11), (12), and (13) is given in Appendix A. The analytical and numerical results are compared in Fig. 9.

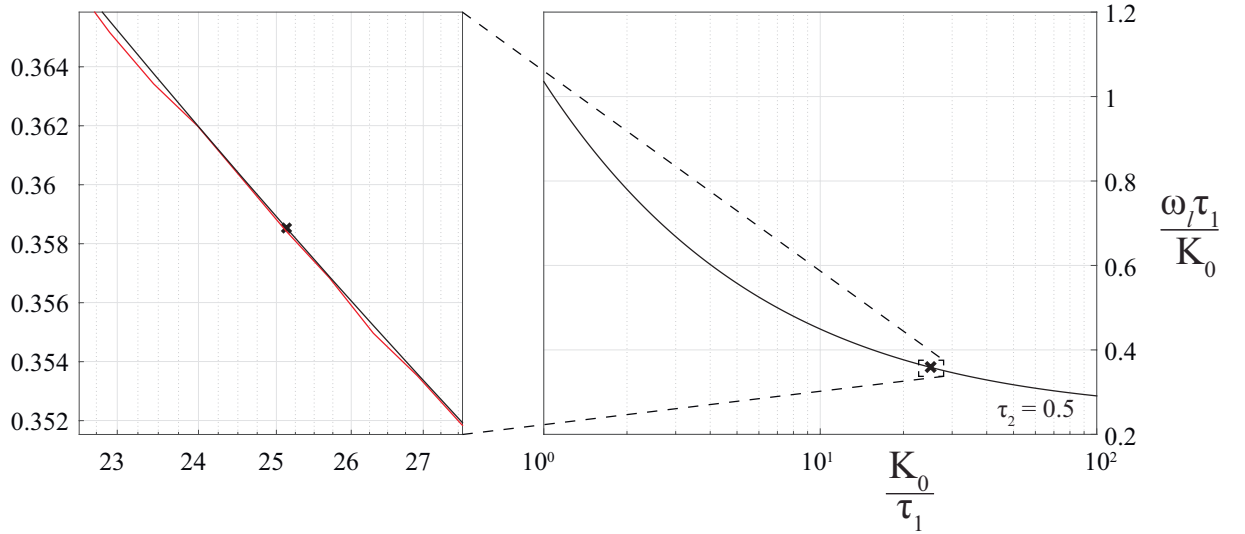


Figure 9: Comparison of analytical and numerical results on the lock-in computation.

4. Conclusion

In the present work the model of PLL with impulse signals and active PI filter in the signal's phase space is described. For the considered PLL the lock-in range is computed analytically and obtained result are compared with numerical simulations.

Appendix A. The lock-in computation

In this section equations (11), (12), and (13) are rigorously derived. Consider the following relations

$$\begin{cases} \dot{\theta}_\Delta = y, \\ \dot{y} = -aK_0 \dot{\varphi}(\theta_\Delta) y - bK_0 \varphi(\theta_\Delta). \end{cases} \quad (\text{A.1})$$

Also we consider a normalized 2π -periodic zigzag function

$$\varphi(\theta_\Delta) = \begin{cases} k\theta_\Delta, & \text{if } -\frac{1}{k} \leq \theta_\Delta \leq \frac{1}{k}; \\ -\frac{k}{\pi k - 1} \theta_\Delta + \frac{\pi k}{\pi k - 1}, & \text{if } \frac{1}{k} \leq \theta_\Delta \leq 2\pi - \frac{1}{k} \end{cases} \quad (\text{A.2})$$

for finite $k > \frac{1}{\pi}$ in the interval $\theta_\Delta \in \left[-\frac{1}{k}, 2\pi - \frac{1}{k}\right)$. For $k = \frac{2}{\pi}$ the function $\varphi(\theta_\Delta)$ is triangular and corresponds to (1).

From 2π -periodicity of (A.1) it follows that for each interval the behavior of phase trajectories on the system phase plane is the same

$$\theta_\Delta \in \left(-\frac{1}{k} + 2\pi j, -\frac{1}{k} + 2\pi(j+1)\right], \quad j \in \mathbb{Z}.$$

Thus, we can consider a single interval $\left(-\frac{1}{k}, 2\pi - \frac{1}{k}\right]$ of the phase plane of (A.1).

In the intervals inside $\left(-\frac{1}{k}, 2\pi - \frac{1}{k}\right]$, (A.1) takes the form:

I. $-\frac{1}{k} < \theta_\Delta < \frac{1}{k}$

$$\begin{cases} \dot{\theta}_\Delta = y, \\ \dot{y} = -aK_0ky - bK_0k\theta_\Delta; \end{cases} \quad (\text{A.3})$$

II. $\frac{1}{k} < \theta_\Delta < 2\pi - \frac{1}{k}$

$$\begin{cases} \dot{\theta}_\Delta = y, \\ \dot{y} = aK_0\frac{k}{\pi k-1}y + bK_0\left(\frac{k}{\pi k-1}\theta_\Delta - \frac{\pi k}{\pi k-1}\right). \end{cases} \quad (\text{A.4})$$

In each interval there exists only one equilibrium:

I. $-\frac{1}{k} < \theta_\Delta < \frac{1}{k}$

$$\begin{cases} y_{\text{eq}} = 0, \\ -aK_0ky - bK_0k\theta_{\text{eq}} = 0; \end{cases} \quad \begin{cases} y_{\text{eq}} = 0, \\ \theta_{\text{eq}} = 0; \end{cases}$$

II. $\frac{1}{k} < \theta_\Delta < 2\pi - \frac{1}{k}$

$$\begin{cases} y_{\text{eq}} = 0, \\ \frac{aK_0k}{\pi k-1}y_{\text{eq}} + \frac{bK_0k}{\pi k-1}(\theta_{\text{eq}} - \pi) = 0. \end{cases} \quad \begin{cases} y_{\text{eq}} = 0, \\ \theta_{\text{eq}} = \pi. \end{cases}$$

To define a type of the equilibria points, we compute the corresponding characteristic polynomial and eigenvalues. For the first equilibrium $(\theta_{\text{eq}}, y_{\text{eq}}) = (0, 0)$ the characteristic polynomial is as follows

$$\chi(\lambda) = \begin{vmatrix} -\lambda & 1 \\ -bK_0k & -aK_0k - \lambda \end{vmatrix} = \lambda^2 + aK_0k\lambda + bK_0k.$$

The eigenvalues of the equilibrium $(\theta_{\text{eq}}, y_{\text{eq}}) = (0, 0)$ depend on a sign of $(aK_0)^2 - \frac{4bK_0}{k}$. Here, there exist three cases:

A. $(aK_0)^2 - \frac{4bK_0}{k} > 0$:

$$\lambda_{1,2}^s = \frac{-aK_0k \pm \sqrt{(aK_0k)^2 - 4bK_0k}}{2},$$

the equilibrium $(0, 0)$ is a stable node.

B. $(aK_0)^2 - \frac{4bK_0}{k} = 0$:

$$\lambda_1^s = \lambda_2^s = \frac{-aK_0k}{2},$$

the equilibrium $(0,0)$ is a stable degenerated node, or stable proper node.

C. $(aK_0)^2 - \frac{4bK_0}{k} < 0$:

$$\lambda_{1,2}^s = \frac{-aK_0k \pm i\sqrt{4bK_0k - (aK_0k)^2}}{2},$$

the equilibrium $(0,0)$ is a stable focus.

Denote $(\theta_{\text{eq}}^s, y_{\text{eq}}) = (0,0)$.

For the second equilibrium $(\theta_{\text{eq}}, y_{\text{eq}}) = (\pi, 0)$ we have

$$\chi(\lambda) = \begin{vmatrix} -\lambda & 1 \\ \frac{bK_0k}{\pi k - 1} & \frac{aK_0k}{\pi k - 1} - \lambda \end{vmatrix} = \lambda^2 - \frac{aK_0k}{\pi k - 1}\lambda - \frac{bK_0k}{\pi k - 1};$$

$$\lambda_{1,2}^u = \frac{\frac{aK_0k}{\pi k - 1} \pm \sqrt{\left(\frac{aK_0k}{\pi k - 1}\right)^2 + \frac{4bK_0k}{\pi k - 1}}}{2},$$

which means that $(\pi, 0)$ is always an unstable saddle for the considered parameters of the PLL. Denote $(\theta_{\text{eq}}^u, y_{\text{eq}}) = (\pi, 0)$.

The calculation of $S'(\theta_{\text{eq}}^s)$ from formula (10) for lock-in range is in some stages. First, find two-dimensional eigenvectors X_1^u, X_2^u of saddle point $(\theta_{\text{eq}}^u, y_{\text{eq}})$ from the interval $\theta_{\Delta} \in \left(\frac{1}{k}, 2\pi - \frac{1}{k}\right)$. Next, compute $S'(\frac{1}{k})$, which is possible due to the continuity of (A.1). Find two-dimensional eigenvectors X_1^s, X_2^s of stable equilibrium $(\theta_{\text{eq}}^s, y_{\text{eq}})$ in the interval $\theta_{\Delta} \in \left(-\frac{1}{k}, \frac{1}{k}\right)$. Find a general solution of (A.1) in the interval $\theta_{\Delta} \in \left(-\frac{1}{k}, \frac{1}{k}\right)$. Using the obtained $S'(\frac{1}{k})$ as the initial data of the Cauchy problem, we can compute $S'(\theta_{\text{eq}}^s)$.

Let us find the eigenvectors X_1^u, X_2^u of a saddle point $(\theta_{\text{eq}}^u, y_{\text{eq}})$. First, find the eigenvector X_1^u :

$$\begin{pmatrix} -\lambda_1^u & 1 \\ \frac{bK_0k}{\pi k - 1} & \frac{aK_0k}{\pi k - 1} - \lambda_1^u \end{pmatrix} X_1^u = \mathbb{O},$$

$$\begin{pmatrix} -\frac{\frac{aK_0k}{\pi k - 1} + \sqrt{\left(\frac{aK_0k}{\pi k - 1}\right)^2 + \frac{4bK_0k}{\pi k - 1}}}{2} & 1 \\ \frac{bK_0k}{\pi k - 1} & \frac{aK_0k}{\pi k - 1} - \frac{\frac{aK_0k}{\pi k - 1} + \sqrt{\left(\frac{aK_0k}{\pi k - 1}\right)^2 + \frac{4bK_0k}{\pi k - 1}}}{2} \end{pmatrix} X_1^u = \mathbb{O},$$

$$\begin{pmatrix} -\frac{\frac{aK_0k}{\pi k - 1} + \sqrt{\left(\frac{aK_0k}{\pi k - 1}\right)^2 + \frac{4bK_0k}{\pi k - 1}}}{2} & 1 \\ \frac{bK_0k}{\pi k - 1} & -\frac{\frac{aK_0k}{\pi k - 1} - \sqrt{\left(\frac{aK_0k}{\pi k - 1}\right)^2 + \frac{4bK_0k}{\pi k - 1}}}{2} \end{pmatrix} X_1^u = \mathbb{O}. \quad (\text{A.5})$$

Multiply the second row of (A.5) by $\frac{\frac{aK_0k}{\pi k - 1} + \sqrt{\left(\frac{aK_0k}{\pi k - 1}\right)^2 + \frac{4bK_0k}{\pi k - 1}}}{2}$ and divide it by $\frac{bK_0k}{\pi k - 1}$.

Then we have

$$\begin{pmatrix} -\frac{\frac{aK_0k}{\pi k-1} + \sqrt{\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}}}{2} & 1 \\ \frac{\frac{aK_0k}{\pi k-1} + \sqrt{\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}}}{2} & -\frac{\left(\left(\frac{aK_0k}{\pi k-1}\right)^2 - \left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}\right)(\pi k-1)}{4bK_0k} \end{pmatrix} X_1^u = \mathbb{O},$$

$$\begin{pmatrix} -\frac{\frac{aK_0k}{\pi k-1} + \sqrt{\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}}}{2} & 1 \\ \frac{\frac{aK_0k}{\pi k-1} + \sqrt{\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}}}{2} & -1 \end{pmatrix} X_1^u = \mathbb{O}.$$

Hence,

$$X_1^u = \begin{pmatrix} c \\ c \frac{\sqrt{(aK_0k)^2 + 4bK_0k(\pi k-1)} + aK_0k}{2(\pi k-1)} \end{pmatrix}.$$

Let us choose $c = \frac{\sqrt{(aK_0k)^2 + 4bK_0k(\pi k-1)} - aK_0k}{2bK_0k}$. Then

$$X_1^u = \begin{pmatrix} \frac{\sqrt{(aK_0k)^2 + 4bK_0k(\pi k-1)} - aK_0k}{2bK_0k} \\ \frac{(aK_0k)^2 + 4bK_0k(\pi k-1) - (aK_0k)^2}{4bK_0k(\pi k-1)} \end{pmatrix},$$

$$X_1^u = \begin{pmatrix} \frac{\sqrt{(aK_0k)^2 + 4bK_0k(\pi k-1)} - aK_0k}{2bK_0k} \\ 1 \end{pmatrix}.$$

Next, find the second eigenvector X_2^u in the same way:

$$\begin{pmatrix} -\lambda_2^u & 1 \\ \frac{bK_0k}{\pi k-1} & \frac{aK_0k}{\pi k-1} - \lambda_2^u \end{pmatrix} X_2^u = \mathbb{O},$$

$$\begin{pmatrix} \frac{\sqrt{\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}} - \frac{aK_0k}{\pi k-1}}{2} & 1 \\ \frac{bK_0k}{\pi k-1} & \frac{aK_0k}{\pi k-1} + \frac{\sqrt{\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}} - \frac{aK_0k}{\pi k-1}}{2} \end{pmatrix} X_2^u = \mathbb{O},$$

$$\begin{pmatrix} \frac{\sqrt{\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}} - \frac{aK_0k}{\pi k-1}}{2} & 1 \\ \frac{bK_0k}{\pi k-1} & \frac{\sqrt{\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}} + \frac{aK_0k}{\pi k-1}}{2} \end{pmatrix} X_2^u = \mathbb{O}. \quad (\text{A.6})$$

Multiply the second row of (A.6) by $\frac{\sqrt{\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}} - \frac{aK_0k}{\pi k-1}}{2}$, and divide it by $\frac{bK_0k}{\pi k-1}$. Then

$$\begin{pmatrix} \frac{\sqrt{\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}} - \frac{aK_0k}{\pi k-1}}{2} & 1 \\ \frac{\sqrt{\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}} - \frac{aK_0k}{\pi k-1}}{2} & \frac{\left(\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1} - \left(\frac{aK_0k}{\pi k-1}\right)^2\right)(\pi k-1)}{4bK_0k} \end{pmatrix} X_2^u = \mathbb{O},$$

$$\begin{pmatrix} \frac{\sqrt{\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}} - \frac{aK_0k}{\pi k-1}}{2} & 1 \\ \frac{\sqrt{\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}} - \frac{aK_0k}{\pi k-1}}{2} & 1 \end{pmatrix} X_2^u = \mathbb{O}.$$

Hence,

$$X_2^u = \begin{pmatrix} -c \\ c \frac{\sqrt{\left(\frac{aK_0k}{\pi k-1}\right)^2 + \frac{4bK_0k}{\pi k-1}} - \frac{aK_0k}{\pi k-1}}{2} \end{pmatrix}.$$

Choose $c = \frac{\sqrt{(aK_0k)^2 + 4bK_0k(\pi k-1)} + aK_0k}{2bK_0k}$:

$$X_2^u = \begin{pmatrix} -\frac{\sqrt{(aK_0k)^2 + 4bK_0k(\pi k-1)} + aK_0k}{2bK_0k} \\ \frac{(aK_0k)^2 + 4bK_0k(\pi k-1) - (aK_0k)^2}{4bK_0k(\pi k-1)} \end{pmatrix},$$

$$X_2^u = \begin{pmatrix} -\frac{\sqrt{(aK_0k)^2 + 4bK_0k(\pi k-1)} + aK_0k}{2bK_0k} \\ 1 \end{pmatrix}.$$

We can show that the direction of separatrix $S'(\theta_\Delta)$ coincides with the direction of eigenvector X_2^u , which corresponds to eigenvalue λ_2^u . That allows us to find $S'(\frac{1}{k})$. For this purpose, we write an equation of straight line, which passes through two points

$$(x_1, y_1) = (\pi, 0),$$

$$(x_2, y_2) = \left(\pi - \frac{\sqrt{(aK_0k)^2 + 4bK_0k(\pi k-1)} + aK_0k}{2bK_0k}, 1 \right).$$

The equation takes the form

$$\begin{aligned}\frac{y-0}{1-0} &= \frac{x-\pi}{\left(\pi - \frac{\sqrt{(aK_0k)^2 + 4bK_0k(\pi k - 1)} + aK_0k}{2bK_0k}\right) - \pi}, \\ y &= \frac{2bK_0k}{\sqrt{(aK_0k)^2 + 4bK_0k(\pi k - 1)} + aK_0k} (\pi - x), \\ y &= \frac{2bK_0k \left(\sqrt{(aK_0k)^2 + 4bK_0k(\pi k - 1)} - aK_0k\right)}{(aK_0k)^2 + 4bK_0k(\pi k - 1) - (aK_0k)^2} (\pi - x), \\ y &= \frac{\sqrt{(aK_0k)^2 + 4bK_0k(\pi k - 1)} - aK_0k}{2(\pi k - 1)} (\pi - x).\end{aligned}$$

Then

$$\begin{aligned}S'\left(\frac{1}{k}\right) &= \frac{\sqrt{(aK_0k)^2 + 4bK_0k(\pi k - 1)} - aK_0k}{2(\pi k - 1)} \left(\pi - \frac{1}{k}\right) = \\ &= \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})} - aK_0}{2}.\end{aligned}$$

Next, we need to find the eigenvectors of equilibrium $(\theta_{\text{eq}}^s, y_{\text{eq}})$ and a general solution of (A.1) in the interval $(-\frac{1}{k}, \frac{1}{k})$. It was shown that for a stable equilibrium $(\theta_{\text{eq}}^s, y_{\text{eq}})$ in the interval $(-\frac{1}{k}, \frac{1}{k})$ there exist three different cases, which depend on a sign of $(aK_0)^2 - \frac{4bK_0}{k}$. The eigenvectors X_1^s and X_2^s are computed in the case of stable focus only. For other cases the computation of X_1^s , X_2^s is similar to that, considered in Appendix A.1.

Appendix A.1. Stable node

This case corresponds to $(aK_0)^2 - \frac{4bK_0}{k} > 0$. Let us find the eigenvectors X_1^s , X_2^s :

$$\begin{aligned}\begin{pmatrix} -\lambda_1^s & 1 \\ -bK_0k & -aK_0k - \lambda_1^s \end{pmatrix} X_1^s &= \mathbb{O}, \\ \begin{pmatrix} \frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2} & 1 \\ -bK_0k & -aK_0k + \frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2} \end{pmatrix} X_1^s &= \mathbb{O}, \\ \begin{pmatrix} \frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2} & 1 \\ -bK_0k & -\frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2} \end{pmatrix} X_1^s &= \mathbb{O}. \quad (\text{A.7})\end{aligned}$$

Multiply the second row of (A.7) by $\frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2}$, and divide it by bK_0k :

$$\begin{pmatrix} \frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2} & 1 \\ -\frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2} & -\frac{(aK_0k)^2 - (aK_0k)^2 + 4bK_0k}{4bK_0k} \end{pmatrix} X_1^s = \mathbb{O},$$

$$\begin{pmatrix} \frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2} & 1 \\ -\frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2} & -1 \end{pmatrix} X_1^s = \mathbb{O},$$

$$X_1^s = \begin{pmatrix} -c \\ c \frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2} \end{pmatrix}.$$

Choose $c = -1$. Then

$$X_1^s = \begin{pmatrix} 1 \\ \frac{\sqrt{(aK_0k)^2 - 4bK_0k} - aK_0k}{2} \end{pmatrix}.$$

Next, find eigenvector X_2^s :

$$\begin{pmatrix} -\lambda_2^s & 1 \\ -bK_0k & -aK_0k - \lambda_2^s \end{pmatrix} X_2^s = \mathbb{O},$$

$$\begin{pmatrix} \frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2} & 1 \\ -bK_0k & -aK_0k + \frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2} \end{pmatrix} X_2^s = \mathbb{O},$$

$$\begin{pmatrix} \frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2} & 1 \\ -bK_0k & \frac{\sqrt{(aK_0k)^2 - 4bK_0k} - aK_0k}{2} \end{pmatrix} X_2^s = \mathbb{O}. \quad (\text{A.8})$$

Multiply the second row of (A.8) by $\frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2}$, and divide it by bK_0k :

$$\begin{pmatrix} \frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2} & 1 \\ -\frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2} & \frac{(aK_0k)^2 - 4bK_0k - (aK_0k)^2}{4bK_0k} \end{pmatrix} X_2^s = \mathbb{O},$$

$$\begin{pmatrix} \frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2} & 1 \\ -\frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2} & -1 \end{pmatrix} X_2^s = \mathbb{O},$$

$$X_2^s = \begin{pmatrix} -c \\ c \frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2} \end{pmatrix}.$$

Choose $c = -1$. Then

$$X_2^s = \begin{pmatrix} 1 \\ -\frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2} \end{pmatrix}.$$

In the interval $\theta_\Delta \in \left(-\frac{1}{k}, \frac{1}{k}\right)$ for $(\theta_{\text{eq}}^s, y_{\text{eq}}) = (0, 0)$ being a node, a general solution of (A.1) has the form:

$$\begin{cases} \theta_\Delta(t) = c_1 e^{\lambda_1^s t} + c_2 e^{\lambda_2^s t}, \\ y(t) = -c_1 \frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2} e^{\lambda_1^s t} - c_2 \frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2} e^{\lambda_2^s t}. \end{cases} \quad (\text{A.9})$$

Let us find coefficients c_1, c_2 of (A.9) for the solution of the Cauchy problem with initial conditions $\theta_\Delta(0) = \frac{1}{k}$, $y(0) = \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})} - aK_0}{2}$, which coincide with $S'(\frac{1}{k})$. At moment $t = 0$ we have

$$\begin{cases} \frac{1}{k} = c_1 + c_2, \\ \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})} - aK_0}{2} = \\ = -c_1 \frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2} - c_2 \frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2}, \end{cases}$$

$$\begin{cases} c_2 = \frac{1}{k} - c_1, \\ \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})} - aK_0}{2} + \frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2k} = \\ = -c_1 \frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2} + c_1 \frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2}, \end{cases}$$

$$\begin{cases} c_2 = \frac{1}{k} - c_1, \\ \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})}}{2} + \frac{\sqrt{(aK_0)^2 - \frac{4bK_0}{k}}}{2} = c_1 k \sqrt{(aK_0)^2 - \frac{4bK_0}{k}}, \end{cases}$$

$$\begin{cases} c_2 = \frac{1}{k} - c_1, \\ c_1 = \left(\frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})}}{\sqrt{(aK_0)^2 - \frac{4bK_0}{k}}} + 1 \right) : 2k, \end{cases}$$

$$\begin{cases} c_1 = \left(\frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})}}{\sqrt{(aK_0)^2 - \frac{4bK_0}{k}}} + 1 \right) : 2k, \\ c_2 = \left(1 - \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})}}{\sqrt{(aK_0)^2 - \frac{4bK_0}{k}}} \right) : 2k. \end{cases} \quad (\text{A.10})$$

Finally, find $y(t_0)$ under the condition $\theta_\Delta(t_0) = 0$. The value of $y(t_0)$ corresponds to $S'(\theta_{\text{eq}}^s)$. For this purpose, we express $y(t_0)$ in terms of c_1, c_2 from (A.10). Then

$$\begin{cases} 0 = c_1 e^{\lambda_1^s t_0} + c_2 e^{\lambda_2^s t_0}, \\ y(t_0) = -c_1 \frac{aK_0 k - \sqrt{(aK_0 k)^2 - 4bK_0 k}}{2} e^{\lambda_1^s t_0} - c_2 \frac{aK_0 k + \sqrt{(aK_0 k)^2 - 4bK_0 k}}{2} e^{\lambda_2^s t_0}, \end{cases}$$

$$\begin{cases} -\frac{c_1}{c_2} = e^{(\lambda_2^s - \lambda_1^s) t_0}, \\ y(t_0) = -c_1 \frac{aK_0 k - \sqrt{(aK_0 k)^2 - 4bK_0 k}}{2} e^{\lambda_1^s t_0} - c_2 \frac{aK_0 k + \sqrt{(aK_0 k)^2 - 4bK_0 k}}{2} e^{\lambda_2^s t_0}, \end{cases}$$

$$\begin{cases} -\frac{c_1}{c_2} = e^{\left(-\frac{\sqrt{(aK_0 k)^2 - 4bK_0 k} + aK_0 k}{2} - \frac{\sqrt{(aK_0 k)^2 - 4bK_0 k} - aK_0 k}{2} \right) t_0}, \\ y(t_0) = -c_1 \frac{aK_0 k - \sqrt{(aK_0 k)^2 - 4bK_0 k}}{2} e^{\lambda_1^s t_0} - c_2 \frac{aK_0 k + \sqrt{(aK_0 k)^2 - 4bK_0 k}}{2} e^{\lambda_2^s t_0}, \end{cases}$$

$$\begin{cases} -\frac{c_1}{c_2} = e^{\left(-\sqrt{(aK_0 k)^2 - 4bK_0 k} \right) t_0}, \\ y(t_0) = -c_1 \frac{aK_0 k - \sqrt{(aK_0 k)^2 - 4bK_0 k}}{2} e^{\lambda_1^s t_0} - c_2 \frac{aK_0 k + \sqrt{(aK_0 k)^2 - 4bK_0 k}}{2} e^{\lambda_2^s t_0}, \end{cases}$$

$$\begin{cases} \ln\left(-\frac{c_1}{c_2}\right) = -\left(\sqrt{(aK_0 k)^2 - 4bK_0 k}\right) t_0, \\ y(t_0) = -c_1 \frac{aK_0 k - \sqrt{(aK_0 k)^2 - 4bK_0 k}}{2} e^{\lambda_1^s t_0} - c_2 \frac{aK_0 k + \sqrt{(aK_0 k)^2 - 4bK_0 k}}{2} e^{\lambda_2^s t_0}, \end{cases}$$

$$\begin{cases} t_0 = \frac{\ln\left(-\frac{c_2}{c_1}\right)}{\sqrt{(aK_0 k)^2 - 4bK_0 k}}, \\ y(t_0) = -c_1 \frac{aK_0 k - \sqrt{(aK_0 k)^2 - 4bK_0 k}}{2} e^{\lambda_1^s t_0} - c_2 \frac{aK_0 k + \sqrt{(aK_0 k)^2 - 4bK_0 k}}{2} e^{\lambda_2^s t_0}, \end{cases}$$

Transform the following expression

$$\begin{aligned}
e^{\lambda_1^s t_0} &= e^{\left(\ln \left(-\frac{c_2}{c_1} \right) \frac{\lambda_1^s}{\sqrt{(aK_0k)^2 - 4bK_0k}} \right)} = e^{\left(\ln \left(-\frac{c_2}{c_1} \right) \frac{\sqrt{(aK_0k)^2 - 4bK_0k} - aK_0k}{2\sqrt{(aK_0k)^2 - 4bK_0k}} \right)} = \\
&= \left(e^{\ln \left(-\frac{c_2}{c_1} \right)} \right)^{\left(\frac{1}{2} - \frac{aK_0k}{2\sqrt{(aK_0k)^2 - 4bK_0k}} \right)} = \left(-\frac{c_2}{c_1} \right)^{\left(\frac{1}{2} - \frac{aK_0k}{2\sqrt{(aK_0k)^2 - 4bK_0k}} \right)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
e^{\lambda_2^s t_0} &= e^{\left(\ln \left(-\frac{c_2}{c_1} \right) \frac{\lambda_2^s}{\sqrt{(aK_0k)^2 - 4bK_0k}} \right)} = e^{\left(\ln \left(-\frac{c_2}{c_1} \right) \frac{\sqrt{(aK_0k)^2 - 4bK_0k} + aK_0k}{2\sqrt{(aK_0k)^2 - 4bK_0k}} \right)} = \\
&= \left(e^{\ln \left(-\frac{c_2}{c_1} \right)} \right)^{-\left(\frac{1}{2} + \frac{aK_0k}{2\sqrt{(aK_0k)^2 - 4bK_0k}} \right)} = \left(-\frac{c_2}{c_1} \right)^{-\left(\frac{1}{2} + \frac{aK_0k}{2\sqrt{(aK_0k)^2 - 4bK_0k}} \right)} = \left(-\frac{c_2}{c_1} \right)^{\left(\frac{1}{2} - \frac{aK_0k}{2\sqrt{(aK_0k)^2 - 4bK_0k}} \right)} - 1.
\end{aligned}$$

Then

$$\begin{aligned}
y(t_0) &= -c_1 \frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2} e^{\lambda_1^s t_0} - c_2 \frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2} e^{\lambda_2^s t_0}, \\
y(t_0) &= -c_1 \frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2} \left(-\frac{c_2}{c_1} \right)^{\left(\frac{1}{2} - \frac{aK_0k}{2\sqrt{(aK_0k)^2 - 4bK_0k}} \right)} - \\
&\quad - c_2 \frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2} \left(-\frac{c_2}{c_1} \right)^{\left(\frac{1}{2} - \frac{aK_0k}{2\sqrt{(aK_0k)^2 - 4bK_0k}} \right)} - 1, \\
y(t_0) &= -c_1 \frac{aK_0k - \sqrt{(aK_0k)^2 - 4bK_0k}}{2} \left(-\frac{c_2}{c_1} \right)^{\left(\frac{1}{2} - \frac{aK_0k}{2\sqrt{(aK_0k)^2 - 4bK_0k}} \right)} + \\
&\quad + c_1 \frac{aK_0k + \sqrt{(aK_0k)^2 - 4bK_0k}}{2} \left(-\frac{c_2}{c_1} \right)^{\left(\frac{1}{2} - \frac{aK_0k}{2\sqrt{(aK_0k)^2 - 4bK_0k}} \right)}.
\end{aligned}$$

As a result, for the case $(aK_0k)^2 - 4bK_0k > 0$, when a stable equilibrium $(\theta_{\text{eq}}^s, y_{\text{eq}})$ is a stable node, $S'(\theta_{\text{eq}}^s)$ can be found from the following formula

$$S'(\theta_{\text{eq}}^s) = c_1 \sqrt{(aK_0k)^2 - 4bK_0k} \left(-\frac{c_2}{c_1} \right)^{\left(\frac{1}{2} - \frac{aK_0k}{2\sqrt{(aK_0k)^2 - 4bK_0k}} \right)}, \quad (\text{A.11})$$

where

$$c_1 = \left(\frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})}}{\sqrt{(aK_0)^2 - \frac{4bK_0}{k}}} + 1 \right) : 2k, \quad c_2 = \left(1 - \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})}}{\sqrt{(aK_0)^2 - \frac{4bK_0}{k}}} \right) : 2k.$$

Appendix A.2. Stable focus

This case corresponds to $(aK_0)^2 - \frac{4bK_0}{k} < 0$. The eigenvectors X_1^s , X_2^s are found in the same way as in Appendix A.1:

$$X_1^s = \begin{pmatrix} 1 \\ -\frac{aK_0k - i\sqrt{4bK_0k - (aK_0k)^2}}{2} \end{pmatrix}, \quad X_2^s = \begin{pmatrix} 1 \\ -\frac{aK_0k + i\sqrt{4bK_0k - (aK_0k)^2}}{2} \end{pmatrix}.$$

The eigenvectors X_1^s , X_2^s can be represented as

$$X_{1,2}^s(t) = U_{1,2}^s + iV_{1,2}^s,$$

where $U_{1,2}^s$, $V_{1,2}^s$ are real two-dimensional vectors:

$$U_1^s = \begin{pmatrix} 1 \\ -\frac{aK_0k}{2} \end{pmatrix}, \quad V_1^s = \begin{pmatrix} 0 \\ \frac{\sqrt{4bK_0k - (aK_0k)^2}}{2} \end{pmatrix},$$

$$U_2^s = \begin{pmatrix} 1 \\ -\frac{aK_0k}{2} \end{pmatrix}, \quad V_2^s = \begin{pmatrix} 0 \\ -\frac{\sqrt{4bK_0k - (aK_0k)^2}}{2} \end{pmatrix}.$$

Consider a solution of (A.1), which corresponds to the eigenvalue λ_1^s :

$$\begin{aligned} Y_1(t) &= e^{\lambda_1^s t} X_1^s = e^{(\text{Re } \lambda_1^s + i \text{Im } \lambda_1^s)t} (U_1^s + iV_1^s) = \\ &= e^{t \text{Re } \lambda_1^s} (\cos(t \text{Im } \lambda_1^s) + i \sin(t \text{Im } \lambda_1^s)) (U_1^s + iV_1^s) = \\ &= e^{t \text{Re } \lambda_1^s} (U_1^s \cos(t \text{Im } \lambda_1^s) - V_1^s \sin(t \text{Im } \lambda_1^s)) + \\ &+ i e^{t \text{Re } \lambda_1^s} (U_1^s \sin(t \text{Im } \lambda_1^s) + V_1^s \cos(t \text{Im } \lambda_1^s)). \end{aligned}$$

A general solution of (A.1) takes the form

$$\begin{aligned} Y(t) &= c_1 e^{t \text{Re } \lambda_1^s} (U_1^s \cos(t \text{Im } \lambda_1^s) - V_1^s \sin(t \text{Im } \lambda_1^s)) + \\ &+ c_2 e^{t \text{Re } \lambda_1^s} (U_1^s \sin(t \text{Im } \lambda_1^s) + V_1^s \cos(t \text{Im } \lambda_1^s)) = \\ &= e^{t \text{Re } \lambda_1^s} U_1^s (c_1 \cos(t \text{Im } \lambda_1^s) + c_2 \sin(t \text{Im } \lambda_1^s)) + \\ &+ e^{t \text{Re } \lambda_1^s} V_1^s (c_2 \cos(t \text{Im } \lambda_1^s) - c_1 \sin(t \text{Im } \lambda_1^s)). \end{aligned}$$

In other words,

$$\begin{cases} \theta_\Delta(t) = e^{t \text{Re } \lambda_1^s} (c_1 \cos(t \text{Im } \lambda_1^s) + c_2 \sin(t \text{Im } \lambda_1^s)), \\ y(t) = -\frac{aK_0k}{2} e^{t \text{Re } \lambda_1^s} (c_1 \cos(t \text{Im } \lambda_1^s) + c_2 \sin(t \text{Im } \lambda_1^s)) + \\ + \frac{e^{t \text{Re } \lambda_1^s} \sqrt{4bK_0k - (aK_0k)^2}}{2} (c_2 \cos(t \text{Im } \lambda_1^s) - c_1 \sin(t \text{Im } \lambda_1^s)). \end{cases} \quad (\text{A.12})$$

Let us find the coefficients c_1, c_2 for the solution of the Cauchy problem with the initial data $\theta_\Delta(0) = \frac{1}{k}$, $y(0) = \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})} - aK_0}{2}$, similarly to Appendix A.1.

$$\begin{cases} \frac{1}{k} = e^{0 \operatorname{Re} \lambda_1^s} (c_1 \cos(0 \operatorname{Im} \lambda_1^s) + c_2 \sin(0 \operatorname{Im} \lambda_1^s)), \\ \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})} - aK_0}{2} = \\ = -\frac{aK_0 k e^{0 \operatorname{Re} \lambda_1^s}}{2} (c_1 \cos(0 \operatorname{Im} \lambda_1^s) + c_2 \sin(0 \operatorname{Im} \lambda_1^s)) + \\ + \frac{e^{0 \operatorname{Re} \lambda_1^s} \sqrt{4bK_0 k - (aK_0 k)^2}}{2} (c_2 \cos(0 \operatorname{Im} \lambda_1^s) - c_1 \sin(0 \operatorname{Im} \lambda_1^s)), \end{cases}$$

$$\begin{cases} \frac{1}{k} = c_1, \\ \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})} - aK_0}{2} = -\frac{aK_0 k}{2} c_1 + \frac{\sqrt{4bK_0 k - (aK_0 k)^2}}{2} c_2, \end{cases}$$

$$\begin{cases} c_1 = \frac{1}{k}, \\ c_2 = \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})}}{k \sqrt{\frac{4bK_0}{k} - (aK_0)^2}}. \end{cases}$$

Next, let us find t_0 such that $\theta_\Delta(t_0) = 0$.

$$\begin{aligned} 0 &= e^{t_0 \operatorname{Re} \lambda_1^s} (c_1 \cos(t_0 \operatorname{Im} \lambda_1^s) + c_2 \sin(t_0 \operatorname{Im} \lambda_1^s)), \\ -\frac{c_1}{c_2} &= \operatorname{tg}(t_0 \operatorname{Im} \lambda_1^s), \\ t_0 &= \frac{\operatorname{arctg}\left(-\frac{c_1}{c_2}\right)}{\operatorname{Im} \lambda_1^s}. \end{aligned}$$

Finally, all the unknowns for $y(t_0)$ from (A.12) are found and $S'(\theta_{\text{eq}}^s)$ is as follows

$$\begin{aligned} S'(\theta_{\text{eq}}^s) &= y(t_0) = -\frac{aK_0 k e^{t_0 \operatorname{Re} \lambda_1^s}}{2} (c_1 \cos(t_0 \operatorname{Im} \lambda_1^s) + c_2 \sin(t_0 \operatorname{Im} \lambda_1^s)) + \\ &+ \frac{e^{t_0 \operatorname{Re} \lambda_1^s} \sqrt{4bK_0 k - (aK_0 k)^2}}{2} (c_2 \cos(t_0 \operatorname{Im} \lambda_1^s) - c_1 \sin(t_0 \operatorname{Im} \lambda_1^s)), \end{aligned} \quad (\text{A.13})$$

where

$$\begin{aligned} t_0 &= \frac{\operatorname{arctg}\left(-\frac{c_1}{c_2}\right)}{\operatorname{Im} \lambda_1^s}, & \lambda_1^s &= \frac{-aK_0 k + i \sqrt{4bK_0 k - (aK_0 k)^2}}{2}, \\ c_1 &= \frac{1}{k}, & c_2 &= \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})}}{k \sqrt{\frac{4bK_0}{k} - (aK_0)^2}}. \end{aligned}$$

Appendix A.3. Stable degenerated node

This case corresponds to $(aK_0)^2 - \frac{4bK_0}{k} = 0$. In this case the eigenvalues λ_1^s and λ_2^s coincide:

$$\lambda^s := -\frac{aK_0k}{2} = \lambda_1^s = \lambda_2^s.$$

A stable equilibrium $(\theta_{\text{eq}}^s, y_{\text{eq}})$ is a stable degenerated node, or a stable proper node.

For the characteristic matrix

$$\begin{pmatrix} -\lambda^s & 1 \\ -bK_0k & -aK_0k - \lambda^s \end{pmatrix}$$

of (A.1) it is shown that $(\theta_{\text{eq}}^s, y_{\text{eq}})$ is a stable degenerated node. Find the eigenvector X^s corresponding to the eigenvalue λ^s of algebraic multiplicity two:

$$\begin{pmatrix} -\lambda^s & 1 \\ -bK_0k & -aK_0k - \lambda^s \end{pmatrix} X^s = \mathbb{O},$$

$$\begin{pmatrix} \frac{aK_0k}{2} & 1 \\ -bK_0k & -\frac{aK_0k}{2} \end{pmatrix} X^s = \mathbb{O}.$$

Adding the first row, multiplied by $\frac{aK_0k}{2}$, to the second row, we have:

$$\begin{pmatrix} \frac{aK_0k}{2} & 1 \\ \frac{(aK_0k)^2}{4} - bK_0k & 0 \end{pmatrix} X^s = \mathbb{O}.$$

The eigenvector X^s can be written as

$$X^s = \begin{pmatrix} c \\ -c\frac{aK_0k}{2} \end{pmatrix}.$$

Choose $c = 1$. To find a general solution of (A.1), we need to additionally find the first associated vector X_1^s :

$$\begin{pmatrix} \frac{aK_0k}{2} & 1 \\ -bK_0k & -\frac{aK_0k}{2} \end{pmatrix} X_1^s = \begin{pmatrix} 1 \\ -\frac{aK_0k}{2} \end{pmatrix},$$

$$\begin{pmatrix} \frac{aK_0k}{2} & 1 \\ \frac{(aK_0k)^2}{4} - bK_0k & 0 \end{pmatrix} X_1^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$X_1^s = \begin{pmatrix} c \\ c - c\frac{aK_0k}{2} \end{pmatrix}.$$

Choose $c = 1$.

A general solution of (A.1) has the form

$$\begin{cases} \theta_{\Delta}(t) = e^{\left(-\frac{aK_0k}{2}t\right)} (c_1 + c_2(t+1)), \\ y(t) = e^{\left(-\frac{aK_0k}{2}t\right)} \left(-\frac{aK_0k}{2}c_1 + c_2\left(-\frac{aK_0k}{2}t + 1 - \frac{aK_0k}{2}\right)\right). \end{cases} \quad (\text{A.14})$$

Similarly to Appendix A.1 and Appendix A.2, let us find the coefficients c_1 and c_2 for the solution of the Cauchy problem with the initial data $\theta_{\Delta}(0) = \frac{1}{k}$ and $y(0) = \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})} - aK_0}{2}$. In this case we have:

$$\begin{cases} \frac{1}{k} = c_1 + c_2, \\ \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})} - aK_0}{2} = -\frac{aK_0k}{2}c_1 + c_2\left(1 - \frac{aK_0k}{2}\right), \end{cases}$$

$$\begin{cases} c_1 = \frac{1}{k} - c_2, \\ \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})} - aK_0}{2} = -\frac{aK_0k}{2}\left(\frac{1}{k} - c_2\right) + c_2 - c_2\frac{aK_0k}{2}, \end{cases}$$

$$\begin{cases} c_1 = \frac{1}{k} - c_2, \\ \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})}}{2} = c_2, \end{cases}$$

$$\begin{cases} c_1 = \frac{1}{k} - \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})}}{2}, \\ c_2 = \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})}}{2}. \end{cases}$$

Find t_0 such that $\theta_{\Delta}(t_0) = 0$:

$$\begin{cases} 0 = e^{\left(-\frac{aK_0k}{2}t_0\right)} (c_1 + c_2(t_0+1)), \\ y(t_0) = e^{\left(-\frac{aK_0k}{2}t_0\right)} \left(-\frac{aK_0k}{2}c_1 + c_2\left(-\frac{aK_0k}{2}t_0 + 1 - \frac{aK_0k}{2}\right)\right), \end{cases}$$

$$\begin{cases} 0 = e^{\left(-\frac{aK_0k}{2}t_0\right)} ((c_1 + c_2) + c_2t_0), \\ y(t_0) = e^{\left(-\frac{aK_0k}{2}t_0\right)} \left(-\frac{aK_0k}{2}(c_1 + c_2) + c_2\left(-\frac{aK_0k}{2}t_0 + 1\right)\right), \end{cases}$$

$$\begin{cases} 0 = \frac{1}{k} + c_2 t_0, \\ y(t_0) = e^{\left(-\frac{aK_0 k}{2} t_0\right)} \left(-\frac{aK_0}{2} + c_2 \left(-\frac{aK_0 k}{2} t_0 + 1\right)\right), \end{cases}$$

$$\begin{cases} t_0 = -\frac{1}{c_2 k}, \\ y(t_0) = e^{\left(-\frac{aK_0 k}{2} t_0\right)} \left(-\frac{aK_0}{2} - \frac{aK_0 k c_2}{2} t_0 + c_2\right). \end{cases}$$

Finally, the expression for $S'(\theta_{\text{eq}}^s)$ is as follows

$$S'(\theta_{\text{eq}}^s) = y(t_0) = c_2 e^{\left(\frac{aK_0}{2c_2}\right)}, \quad (\text{A.15})$$

where

$$c_2 = \frac{\sqrt{(aK_0)^2 + 4bK_0(\pi - \frac{1}{k})}}{2}.$$

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